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Lifschitz Singularities for Periodic Operators Plus Random Potentials

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Using χ-bounding (lower bounds by Laplacians with mixed boundary conditions and discrete analogs), we obtain the Lifschitz exponent at the bottom of the spectrum for random operators of type $H_\omega = T + V_\omega$, with T a (periodic) generator of a positivity-preserving semigroup, extending results by Kirsch and Simon.

KEY WORDS: Random operators; density of states; Lifschitz singularity; χ-bounding; Dirichlet forms.

1. INTRODUCTION AND DISCUSSION

Recently Kirsch and Simon[6] considered the asymptotic behavior near the bottom of the spectrum of the integrated density of states $k(E)$ of the periodic plus random Schrödinger operators

$$H_\omega = T + V_\omega \tag{1}$$

where the "kinetic" energy operator incorporates a periodic potential:

$$T = -\Delta + U_{\text{per}} \tag{2}$$

with $U_{\text{per}}(x+a) = U_{\text{per}}(x)$, $a \in \mathbb{Z}^d$, $U_{\text{per}} \in L^p_{\text{loc}}\mathbb{R}^d)$ with $p=2$ for $d \leqslant 3$, $p = 2+\varepsilon$ for $d=4$, and $p = d/2$ for $d \geqslant 5$. The random potential mimics a distribution of impurities in the crystal described by T:

$$V_\omega(x) = \sum_{i \in \mathbb{Z}^d} q_i(\omega)\, u(x-i) \tag{3}$$

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where the source potential $u(x) \geqslant 0$, and for large $|x|$ is either rapidly decreasing or

$$u(x) = O(|x|^{-d-\alpha}) \tag{4}$$

for some $\alpha > 0$. One has $u \in L^p(\mathbb{R}^d)$ with p as above. The $q_i(\omega)$ are independent, identically distributed random variables. Their distribution function μ has compact support, $\mu[a] < 1$, $a = \min \operatorname{supp} \mu$, and $\mu[a, a+\varepsilon] = O(\varepsilon^\delta)$ for some $\delta \geqslant 0$.

In our case, the relevant definition of the integrated density of states is

$$k(E) = \lim_{\Lambda \uparrow \mathbb{R}^d} |\Lambda|^{-1} \mathcal{N}(E, H_\omega^\Lambda) \tag{5}$$

where H_ω^Λ is any restriction (defined by boundary conditions) of a typical realization of H to the compact domain Λ; $\mathcal{N}(E, X)$ is the number of the eigenvalues of the operator X that are less than E.

Another interesting class of random operators are the finite-difference ones on $l^2(\mathbb{Z}^d)$, which are given by the sum, Eq. (1), of a periodic "kinetic energy" T,

$$(Tf)(n) = -\sum_{n \in \mathbb{Z}^d} I(n, m) f(m) \tag{6}$$

with periodic coefficients

$$I(n+a, m+a) = I(n, m) \tag{7}$$

$\forall a = Kb$, $b \in \mathbb{Z}^d$ for some $K \in \mathbb{N}$, and with V_ω a multiplicative operator given by Eq. (3) with x replaced by n.

The main result of this paper is the following:

Theorem 1. Let H_ω, Eq. (1), be either a continuum Schrödinger operator on $L^2(\mathbb{R}^d)$ satisfying the conditions (2)–(4), or a discrete operator on $l^2(\mathbb{Z}^d)$, Eqs. (6), (7), (3)–(4), with nonnegative $I(n, m)$ of finite range $I(n, m) = 0$, $|n - m| > A$, and $\min_{|n-m|=1} I(n, m) > 0$. Then $k(E)$ has a Lifschitz singularity at $E_0 = \min \operatorname{Sp} H_\omega$:

$$\lim_{E \downarrow E_0} \frac{\ln |\ln k(E)|}{\ln(E - E_0)} = -\lambda \tag{8}$$

where the Lifschitz exponent is

$$\lambda = d/\min(2, \alpha) \tag{9}$$

Remarks. 1. The restriction to $\mathbb{Z}^d$ is only for simplicity of notation; other periodic lattices may be considered. As will be apparent from the proof, in the discrete case the conditions of finite range and nearest neighbor connectedness on $I(n, m)$ can be somewhat relaxed.

2. It is perhaps worth mentioning that all proofs of Lifschitz singularity[1-3,5,6,9-12,16-18,20-22] are for the lowest spectral edge of generators of positivity-preserving semigroups, or the problem may be mapped, as in Ref. 12, onto one with this property. Hopefully, this apparent restriction is technical, due to the difficulty of characterizing the other fluctuative spectral edges.[8] The most interesting cases for applications, such as internal edges for continuum operators, remain unproven.

Kirsch and Simon[6] proved Theorem 1 in the continuum case under the additional restriction that either $d=1$ or the periodic potential is inversion-invariant:

$$U_{\text{per}}(x_1,..., -x_j,..., x_d) = U_{\text{per}}(x_1,..., x_j,..., x_j)$$

for all $1 \leqslant j \leqslant d$.

The dependence of the Lifschitz exponent on α for slowly decaying impurity potentials, Eq. (9), appeared in the work of Pastur,[18] who obtained the leading term of the asymptotic expansion of $k(E)$ as $E \downarrow 0$ for $U_{\text{per}} = 0$ and a Poisson distribution of scatterers with smooth u. This can be understood with the help of a nonrigorous argument patterned on the original one by Lifschitz.[8]

Let $E_0 \in \text{Sp}\, H_\omega$ be a fluctuative spectral edge of H, i.e., $\text{Sp}\, H_\omega$ fills one side of a neighborhood of E_0 where it is pure point and the eigenfunctions reside on local large deviations from the average configuration of the potential. Assume that E_0 is a generalized eigenvalue of T ($Tf_0 = E_0 f_0$ has a (polynomially) bounded solution). Let $f_E^{\omega_0}$ be an eigenfunction of a typical realization ω_0 of H_ω:

$$H_{\omega_0} f_E^{\omega_0} = E f_E^{\omega_0} \tag{10}$$

with $E = E_0 + \eta$. For small $|\eta|$ assume that $f_E^{\omega_0} = \varphi \tilde{f}_0$, where the envelope function φ is slowly varying inside and decays exponentially outside the domain of localization Ω. The function $\tilde{f}_0(x) = f_0(x + y(x))$, with $y(x)$, the "phase modulation" function,[2] slowly varying in Ω. Then $y_{j,k} = \partial y_j / \partial x_h = O(|\Omega|^{-1/d})$ and we can estimate the excess "kinetic energy" by

$$\langle T - E_0 \rangle = (f_E^{\omega_0}, (T - E_0) f_E^{\omega_0}) = c\eta \approx \text{const} \cdot |\Omega|^{-2/d}$$

We may decompose $\langle V_{\omega_0} \rangle = (1 - c)\eta$ into a contribution from the impurities in the domain of localization Ω and one from those outside.

[2] The "phase modulation" is needed to compensate the "phase" mismatch between f_0 and $f_E^{\omega_0}$, which are both rapidly varying and have many zeros for general edges. There is some numerical evidence[15] for long-range phase correlations.

Since outside Ω the situation is "average," the outside contribution may be significant only for power-law-decaying impurity potentials and $\langle V_{\text{out}} \rangle \simeq \text{const} \cdot |\Omega|^{-\alpha/d}$ [with α defined by Eq. (4)]. We see that, assuming comparable constants, $\langle T - E_0 \rangle + \langle V_{\text{out}} \rangle \approx \text{const} \cdot |\Omega|^{-1/\lambda}$, with λ the Lifschitz exponent defined by Eq. (9) and

$$\eta \simeq \text{const} \cdot |\Omega|^{-1/\lambda} + \langle V_{\text{in}} \rangle \tag{11}$$

Let $P_0(\Omega, \eta) = \text{Prob}\{$the sources $q_i(\omega)$, $i \in \Omega$, are such that V_{in} is localizing and $|\langle V_{\text{in}} \rangle| < |\eta|\}$. Then, from Eqs. (5) and (11) we can estimate

$$|k(E_0 + \eta) - k(E_0)| \simeq \max_{\substack{\Omega \\ |\xi| < |\eta|}} P_0(\Omega, \xi)$$

Now we may reasonably expect that $\ln P_0(\Omega, \xi)$ is extensive for large Ω and small $|\xi|$: $\ln P_0(\Omega, \xi) \approx -|\Omega| f(\xi)$, with $f(\xi)$ having at most a logarithmic singularity for $\xi \to 0$.[11] Then,

$$\ln |k(E_0 + \eta) - k(E_0)| \approx -\kappa(\eta)/|\eta|^{\lambda} \tag{12}$$

$E_0 + \eta \in \text{Sp}\, H_\omega$ with $\kappa(\eta)$ having at most a logarithmic singularity. This type of argument can also be used to glean some information on the shape of the Lifschitz tail beyond its tip at E_0.[13]

The recent proofs of Lifschitz singularity for various models[5,6,11,12,20,21] can be regarded as rigorous implementations of the Lifschitz argument by partitioning $\mathbb{R}^d (\mathbb{Z}^d)$ into (congruent) nonoverlapping domains and bracketing H_ω by direct sums of (nearly) statistically independent domain operators

$$\bigoplus_a H_\omega^{\Lambda_{a,l}} \leqslant H_\omega \leqslant \bigoplus_a H_\omega^{\Lambda_{a,u}} \tag{13}$$

whose integrated densities of states will bracket (in reverse order) $k(E)$. Since the simplest way of constructing domain operators with the required properties is Dirichlet–Neumann bracketing (or its analog for finite-difference operators) this method was chosen in practice.

As is apparent from Kirsch and Simon[6] [who proved Eq. (8), with lim replaced by lim sup and = by $\leqslant$, without restrictions on U_{per} in the continuum case], Neumann bounding does not work well for our problem. Replacing it with χ-bounding, which uses suitably adjusted mixed boundary condition Laplacians instead of the Neumann ones, in the continuum case, and an analogous construction in the discrete case, we shall use the same Dirichlet form machinery for the eigenvalue estimates. In the discrete case we shall need an analog of the Dirichlet form, which will be introduced in the next section. Since our proof follows closely the strategy of Kirsch and Simon, only the generalization of some propositions dealing with eigenvalue estimations from that paper will be given; see Ref. 6 for the probabilistic estimates, which remain the same. In the rest of this section we shall define the χ-bounding.

Let $\Lambda \subset \mathbb{R}^d$ be a *convex polyhedron* and $S(\Lambda)$ its boundary.[3] One can define an extension operator that maps any element f of the Sobolev space

$$W_1^2(\Lambda) = \left\{ f \in L^2(\Lambda) \colon \|f\|_{W_1^2(\Lambda)}^2 = \int_\Lambda (|\nabla f|^2 + |f|^2)\, dx < \infty \right\}$$

into $W_1^2 = W_1^2(\mathbb{R}^d)$, with,[(23)]

$$\|Ef\|_{W_1^2} \leqslant A\, \|f\|_{W_1^2(\Lambda)}$$

Thus, $W_1^2(\Lambda)$ coincides with the space of restrictions to Λ of the elements of W_1^2. Let $f \in W_1^2(\Lambda)$; then its restriction to a hyperplane $(\mathbb{R}^{d-1})$ belongs to $L^2(\mathbb{R}^{d-1})$,[23]

$$\|f \upharpoonright\|_{L^2(\mathbb{R}^{d-1})} \leqslant B\, \|f\|_{W_1^2(\Lambda)}$$

Then, for any *real* $\chi \in L^\infty(S(\Lambda))$, the quadratic form

$$Q_{\Lambda,\chi}(f) = \int_\Lambda |\nabla f(x)|^2\, dx + \int_{S(\Lambda)} \chi(x)\, |f(x)|^2\, dS \tag{14}$$

is semibounded,

$$Q_{\Lambda,\chi}(f) \geqslant -c\, \|\chi\|_{L^\infty(S(\Lambda))}\, \|f\|_{W_1^2(\Lambda)}^2$$

and closed with $W_1^2(\Lambda)$ as the form domain. By standard results,[(19)] $Q_{\Lambda,\chi}$ defines a unique self-adjoint operator: $-\Delta^{\Lambda,\chi}$, the Laplacian with mixed (χ-) boundary conditions on $S(\Lambda)$.

Remarks. 1. The Neumann Laplacian $\Delta^{\Lambda,\mathrm{N}}$ coincides with $\Delta^{\Lambda,\chi}$ for $\chi = 0$; the Dirichlet Laplacian $\Delta^{\Lambda,\mathrm{D}}$ can be obtained by taking the limit $\chi(x) \uparrow \infty$ a.e. on $S(\Lambda)$.

2. Using stronger restriction and embedding theorems,[(23)] one can define $\Delta^{\Lambda,\chi}$ for $\chi_- = \frac{1}{2}(|\chi| - \chi) \in L^q(S(\Lambda))$ with $q > 1$ for $d = 2$ and $q \geqslant d-1$ for $d \geqslant 3$.[(14)]

A useful property of these operators is the possibility to bound them from below in the sense of forms by direct sums of operators of the same type with suitably chosen χ.

[3] This is a particular example of domain with *minimally smooth boundary* (Ref. 23, Chapter VI), for which the extension operator can be constructed.

Proposition 1. (χ-*bounding*). Let the plane Σ divide the convex polyhedron $\Lambda \subset \mathbb{R}^d$ into two convex polyhedra, $\bar{\Lambda}_1 \cap \bar{\Lambda}_2 = \Sigma$. Let $\varphi \in L^\infty(\Sigma)$. Then for any $\chi \in L^\infty(S(\Lambda))$,

$$-\Delta^{\Lambda,\chi} \geqslant -\Delta^{\Lambda_1,\chi_1} \oplus \Delta^{\Lambda_2,\chi_2} \tag{15}$$

in the sense of forms. Here

$$\chi_i(x) = \begin{cases} \chi(x), & x \in S(\Lambda_i) \cap S(\Lambda) \\ (-1)^i \varphi(x), & x \in \Sigma \end{cases} \qquad (i=1,2) \tag{16}$$

Proof. The inclusion relation for the form domains is obvious. Let $f = W_1^2(\Lambda)$. Then,

$$\begin{aligned} Q_{\Lambda,\chi}(f) &= \left(\int_{\Lambda_1} + \int_{\Lambda_2}\right) |\nabla f|^2 \, dx \\ &\quad + \left(\int_{S(\Lambda_1) \cap S(\Lambda)} + \int_{S(\Lambda_2) \cap S(\Lambda)}\right) \chi \, |f|^2 \, dS \\ &= Q_{\Lambda_1,\chi_1}(f \restriction_{\Lambda_1}) + Q_{\Lambda_2,\chi_2}(f \restriction_{\Lambda_2}) \end{aligned}$$

The last equality follows by adding and subtracting $\int_\Sigma \varphi \, |f|^2 \, dS$ and collecting the terms that make up the two quadratic forms. ▮

For discrete operators we use the obvious inequality

$$\begin{pmatrix} 0 & a \\ \bar{a} & 0 \end{pmatrix} \leqslant \begin{pmatrix} |a| \, \chi & 0 \\ 0 & |a| \, \chi^{-1} \end{pmatrix}$$

which is valid for any $0 < \chi < \infty$. Let $\Lambda \subset \mathbb{Z}^d$ and the discrete operator T, Eq. (6). The boundary of Λ is $S(\Lambda) = \{n \in \Lambda : \exists m \in \mathbb{Z}^d \backslash \Lambda, \, I(n, m) \neq 0\}$. Note that $S(\Lambda)$ depends on T, but we shall not clutter the notation. Let P_Λ be the characteristic function of Λ. Let $T^{\Lambda,P} = P_\Lambda T P_\Lambda$ be the restriction of T to functions with supp $f \subset \Lambda$. We define for each finite function χ with supp $\chi \supset S(\Lambda)$

$$T^{\Lambda,\chi} = T^{\Lambda,P} + K_\chi^{S(\Lambda)} \tag{17}$$

where the boundary operator associated with χ is given by

$$(K_\chi^{S(\Lambda)} f)(n) = \chi(n) \, f(n) \tag{18}$$

Proposition 2 (χ-*bracketing*). Let $\mathbb{Z}^d \supset \Lambda = \Lambda_1 \cup \Lambda_2$, $\Lambda_1 \cap \Lambda_2 = \varnothing$, and let $T^{\Lambda,\chi}$ be given by Eqs. (6), (17), and (18). Then, for any φ defined on $S(\Lambda_1) \cup S(\Lambda_2)$ with values in $\mathbb{R}_+$,

$$T^{\Lambda_1,\chi_1^-} \oplus T^{\Lambda_2,\chi_2^-} \leqslant T^{\Lambda,\chi} \leqslant T^{\Lambda_1,\chi_1^+} \oplus T^{\Lambda_2,\chi_2^+} \tag{19}$$

where

$$\chi_i^{\pm}(n) = \chi(n) \pm \sum_{m \in S(\Lambda_{3-i})} |I(n, m)| \, \varphi(m)^{1-2i} \tag{20}$$

for $n \in S(\Lambda_i)$, $i = 1, 2$.

Remark. The discrete case is simpler. Since we do not have domain problems, upper bounds are also possible. The discrete Dirichlet and Neumann bracketing of Refs. 20 and 11 is the particular case when T is the finite-difference Laplacian and $\varphi = 1$.

2. SKETCH OF PROOF

The main difference from the case of short-range source potentials lies in the fact that the domain operators are no longer statistically independent and the contribution of neighboring domains may even dominate that of the "kinetic" energy. For the large deviations that are the only contributors to $k(E)$ near E_0 this poses no problem, since they are practically independent.

As noted above, for the continuum case the lim sup half is already proven. In the discrete case we need only to generalize Proposition 5 of Kirsch and Simon.

The main problem with the generalizations to the discrete case is the difficulty of characterizing and working with the (non l^2) generalized eigenfunctions of T, which correspond to the edge. Generally speaking, these are not positive and there may be several of them (degenerancy). This precludes the use of the nice apparatus of Dirichlet forms. The only case in which we could define a positive generalization of the Dirichlet form to the discrete case is when $I(n, m) \geqslant 0$, i.e., T(and H_ω) generates a positivity-preserving semigroup.

Before proceeding further, we need to define another type of restriction of T to cubes with sides commensurate with the periodicity cell.[4] Let $\Lambda_L = \{m \in \mathbb{Z}^d\colon\ 0 \leqslant m_j \leqslant KL - 1,\ j = 1, 2, \ldots, d\}$ and for each $n \in \mathbb{Z}^d$ let $n(\text{mod } L) \in \Lambda_L$ be the corresponding element with coordinates reduced modulo KL. The *periodic restriction* of T, Eqs. (6), (7), to Λ_L is defined by

$$(T_{\text{per}}^L f)(n) = -\sum_{m \in \Lambda_L} I_L(n, m) f(m) \tag{21}$$

with $n \in \Lambda_L$, and

$$I_L(n, m) = \sum_{k(\text{mod } L) = m} I(n, k) \tag{22}$$

[4] Or congruent lattice tiling domains, like inflations of the Wigner–Seitz cell for lattices other than $\mathbb{Z}^d$.

Let E_0 be the lowest eigenvalue of T^1_{per}:

$$T^1_{\text{per}} h_1 = E_0 h_1 \tag{23}$$

If $I(n, m)$ satisfies the assumptions of Theorem 1, then, by the Perron–Frobenius theorem, E_0 is nondegenerate and $h_1(n) > 0$. Let h be the periodic continuation of h_1. Then, $P_{\Lambda_L} h = h_L > 0$ is the unique ground state of T^L_{per}, with the same ground-state energy E_0. Indeed, replacing f by h in Eq. (21), its rhs will coincide, by periodicity of h, with the lhs of Eq. (23). Thus,

$$\min \operatorname{Sp} T = E_0$$

Since $h(n) > 0$, we may associate to each $f \in l^2(\mathbb{Z}^d)$ an "envelope" function $\varphi \in l^2(\mathbb{Z}^d; h^2)$,

$$\varphi(n) = f(n)/h(n) \tag{24}$$

and define *the Dirichlet form* of T by[5]

$$\mathscr{D}(\varphi) = (f, (T - E_0) f) = \frac{1}{2} \sum_{n,m} I(n, m)\, h(n)\, h(m)\, |\varphi(n) - \varphi(m)|^2 \tag{25}$$

Proposition 3. (Proposition 5 of Ref. 6). Let T be a finite-difference, self-adjoint periodic operator on $l^2(\mathbb{Z}^d)$, Eqs. (6), (7), with non-negative and connected $I(n, m)$. Then, for any nonnegative χ, the lowest eigenvalue of the restriction $T^{\Lambda_L, \chi}$ satisfies

$$\lambda_1(T^{\Lambda_L, \chi}) \geqslant \inf \operatorname{Sp} T + cL^{-2} \tag{26}$$

Proof. Let v be a C_0^∞ function with $\operatorname{supp} v \subset B_{1/3}$ (the ball in $\mathbb{R}^d$ with radius $= 1/3$) and $v(x) = 1$, $x \in B_{1/4}$. Take $f = \varphi h$ with $\varphi(n) = v(n/KL)$ as trial function: $\lambda_1(T^{\Lambda_L, \chi}) - E_0 \leqslant \mathscr{D}(\varphi)\, \|f\|^{-2}$. But,

$$\mathscr{D}(\varphi) \leqslant \text{const} \cdot AL^{d-2} \max_n h^2(n)$$

$$\|f\|^2 \geqslant \text{const} \cdot L^d \sum_{n \in \Lambda_1} h^2(n) \quad \blacksquare$$

Let Λ_L^a, $a \in \mathbb{Z}^d$, be the translation by KLa of Λ_L. Let χ_L^a be given by

$$\chi_L^a = -\frac{1}{h} \frac{\partial h}{\partial n} \bigg|_{S(\Lambda_L^a)} \tag{27}$$

[5] To obtain Eq. (25), we need $I(n, m)$ to be *real* and, of course, $h(n) \neq 0$.

in the continuum case and

$$\chi_a^L(n) = -\sum_{\substack{m = n (\mathrm{mod}\ L) \\ n = m}} I(n, m) \frac{h(m)}{h(n)} \tag{28}$$

in the discrete case. By Proposition 1 (resp. Proposition 2), we can bound T from below by

$$T \geqslant \bigoplus_a T^{\Lambda_L^a, \chi_L^a} \tag{29}$$

since in the continuum case, Eq. (27), $\chi_L^a + \chi_L^b = 0$ if $|a-b| = 1$ and, respectively, has the form required by Eq. (20). In the one-dimensional and reflection-invariant cases considered by Kirsch and Simon this reduces to $\chi_L^a = 0$, i.e., to Neumann bounding. Replacing in their proof the Neumann bounds by Eq. (29), we need the following result.

Proposition 4 (Proposition 2 of Ref. 6). Let $T^{\Lambda_L, \chi}$ (Λ_L a cube of side L) be either a Schrödinger operator on $L^2(\Lambda_L)$, Eq. (3), or as in Proposition 3. Let $h > 0$ be the eigenfunction corresponding to $\lambda_1(T)$. Then the splitting between the first two eigenvalues of $T^{\Lambda_L, \chi}$ satisfies

$$\delta = \lambda_2(T^{\Lambda_L, \chi}) - \lambda_1(T^{\Lambda_L, \chi}) \geqslant cL^{-2} \tag{30}$$

Remark. We could have used the comparison theorem for the gap between the first two eigenvalues[7] for proving Proposition 4. The method given below can also be used for proving a similar comparison theorem for $\lambda_n - \lambda_1$.[14]

Proof. By the minmax principle

$$\lambda_2(T^{\Lambda_L, \chi}) = \sup_g \inf_{\substack{f \in D(T) \\ (g, f) = 0}} \frac{(f, T^{\Lambda_L, \chi} f)}{(f, f)}$$

We introduce the envelope functions $g = \gamma h$, $f = \varphi h$,

$$\delta = \sup_\gamma \inf_{(\gamma, \varphi)_h = 0} \frac{\mathscr{D}(\varphi)}{(\varphi, \varphi)_h} \tag{31}$$

where $(\cdot, \cdot)$ is the scalar product in $L^2(\Lambda_L; h^2\, dx)$, respectively $l^2(\Lambda_L; h^2)$, and $\mathscr{D}(\varphi)$ is the Dirichlet form of $T^{\Lambda_L, \chi}$ given by $\mathscr{D}(\varphi) = \int_{\Lambda_L} h^2\, |\nabla \varphi|^2\, dx$ in the continuum case and by Eq. (25) for the discrete one. Let $\sup h(x) = h_+$, $\inf h(x) = h_-$. The inf may be taken now over functions in

$$\tilde{D} = \{\varphi \in L^2(\Lambda; h^2\, dx) \colon (\nabla)\, \nabla \varphi \in L^2(\Lambda_L; h^2\, dx),\ \partial \varphi / \partial n \restriction_{S(\Lambda_L)} = 0\}$$

for the continuum case and $l^2(\Lambda_L; h^2)$ for the discrete case. But $\mathscr{D}(\varphi) \geqslant h_-^2 \mathscr{D}_0(\varphi)$, $\|\varphi\|_h^2 \leqslant h_+^2 \|\varphi\|^2$, where $\mathscr{D}_0(\varphi)$ is the Dirichlet form for $h(x) = 1$, $x \in \Lambda_L$. Then, taking $\gamma = \gamma_0 = 1/h^2$, we obtain

$$\delta \geqslant \left(\frac{h_-}{h_+}\right)^2 \inf_{\int_{\Lambda_L} \varphi dx = 0} \frac{\int |\nabla\varphi|^2 \, dx}{\int_{\Lambda_L} |\varphi|^2 \, dx} = \left(\frac{\pi h_-}{L h_+}\right)^2$$

in the continuum case. For the discrete case one can replace $I(n, m)$ with $\min_{|n-m|=1} I(n, m)$ for $|n-m| = 1$ and 0 otherwise. ▌

REFERENCES

1. M. M. Benderskii and L. A. Pastur, *Mat, Sb.* **82**:245 (1970).
2. R. Friedberg and J. M. Luttinger, *Phys. Rev. B* **12**:(1975).
3. M. Fukushima, *Osaka J. Math.* **11**:73 (1974).
4. A. B. Harris, *Phys. Rev. B* **8**:3661 (1973).
5. W. Kirsch and F. Martinelli, *Commun. Math. Phys.* **89**:27 (1983).
6. W. Kirsch and B. Simon, *J. Stat. Phys.* **42**:799 (1986).
7. W. Kirsch and B. Simon, *J. Funct. Anal.*, to appear.
8. I. M. Lifschitz, *Adv. Phys.* **13**:483 (1964).
9. I. M. Lifschitz, S. A. Gredeskul, and L. A. Pastur, *Introduction to the Theory of Disordered Systems*, (Nauka, Moscow, 1982) (in Russian).
10. J. M. Luttinger and R. Tao, *Ann. Phys. (NY)* **145**:185 (1983).
11. G. A. Mezincescu, *Phys. Rev. B* **32**:6272 (1985).
12. G. A. Mezincescu, *Commun. Math. Phys.* **103**:167 (1986).
13. G. A. Mezincescu, On the shape of Lifschitz tails, in preparation.
14. G. A. Mezincescu, In preparation.
15. I. Moullet, K. Maschke, and H. Kunz, *J. Noncryst. Solids* **77/78**:17 (1985).
16. S. Nakao, *Jpn. J. Math.* **3**:111 (1977).
17. L. A. Pastur, *Teor. Mat. Fiz.* **32**:88 (1977) (in Russian).
18. H. Nagai, *Osaka J. Math.* **14**:111 (1977).
19. M. Reed and B. Simon, *Methods of Modern Mathematical Physics*, Vol. IV, *Analysis of Operators* (Academic Press, New York, 1977).
20. B. Simon, *J. Stat. Phys.* **38**:65 (1985).
21. B. Simon, *J. Stat. Phys.* **46**:911 (1987).
22. B. Simon, Regularity of the density of states for stochastic Jacobi matrices: A review, Caltech preprint (1986).
23. E. M. Stein, *Singular Integrals and Differentiability Properties of Functions* (Princeton, 1970).